

# FIXED POINT THEOREM FOR DISCONTINUOUS MAPPINGS ON PN SPACES

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ABSTRACT. We present a study on strong t-continuity and measure of discontinuity on PN spaces. As an application, we prove a fixed point theorem for a discontinuous self mapping on PN spaces by means of measure of discontinuity.

## 1. INTRODUCTION

The notion of probabilistic normed space (briefly, a PN space) was introduced by Serstnev. The theory of PN spaces has been dormant after its initial application until it was recently put on new basis by Alsina, Schweizer and Sklar [1]. In 1997, B. L. Guillen et al further studied some classes of probabilistic normed spaces. In this paper, the author has considered Serstnev spaces and obtained some results using minimum t-norms. The results in this paper would be much improved if they were proved for arbitrary t-norms. Recently results about bounded linear operators and continuous operators in PN spaces have been studied using arbitrary t-norms [9].

A study on fixed points of contraction mappings on probabilistic metric spaces (PM spaces) was initiated by V. M. Sehgal and A. T. Bharucha-Reid [12] and later developed by numerous authors, as can be realized upon consulting the list of references given at [3] and [8]. On the other hand, O. Hadzic [7] initiated fixed point theory in random normed spaces. Recently, in [10], the authors have studied approximate fixed point theorems on PM and PN spaces respectively which generalizes the results obtained in [2] and [14].

The purpose of this paper is to prove the existence of fixed points for discontinuous self mappings in PN spaces, a study which carried out by L. J. Cromme and I. Diener [4] on normed spaces.

In the sequel we generally follow the notation and terminology of [11].

## 2. PRELIMINARIES

A *distance distribution function* (briefly, a d.d.f.) is a function  $F$  from the extended positive real line  $\overline{\mathbb{R}} = [0, +\infty]$  into the unit interval  $I = [0, 1]$  that is

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nondecreasing and satisfies  $F(0) = 0$ ,  $F(+\infty) = 1$ . We normalize all d.d.f.'s to be left-continuous on the unextended positive real line  $\mathbb{R} = ]0, +\infty[$ . Particularly, for every  $a$  in  $[0, +\infty[$ , the functions  $\varepsilon_a$  is d.d.f. defined by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a; \\ 1, & x > a. \end{cases}$$

The set of all d.d.f.'s will be denoted by  $\Delta^+$ . The set  $\Delta^+$  is partially ordered by the usual pointwise partial ordering of functions, i.e.,  $F \leq G$  whenever  $F(x) \leq G(x)$ , for all  $x$  in  $\mathbb{R}$ . The maximal element for  $\Delta^+$  in this order is the d.d.f.  $\varepsilon_0$ .

A sequence  $\{F_n\}$  in  $\Delta^+$  is said to be weakly convergent to  $F \in \Delta^+$  (shortly,  $F_n \xrightarrow{w} F$ ) if  $\lim_{n \rightarrow \infty} F_n(x) = F(x)$  for every continuity point  $x$  of  $F$ .

A *triangle function* is a binary operation on  $\Delta^+$  that is commutative, associative, nondecreasing in each place and has  $\varepsilon_0$  as identity. Continuity of a triangle function means uniform continuity with respect to the natural product topology on  $\Delta^+ \times \Delta^+$ .

The typical triangle function is the operation  $\tau_T$  which given by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v))$$

for all  $F, G \in \Delta^+$  and all  $x > 0$  ([11], section 7.2 and 7.3). Here  $T$  is a *t-norm*, i.e.,  $T$  is binary operation on  $[0, 1]$  that is commutative, associative, nondecreasing in each place and has 1 as identity.

The most important t-norms are the function  $W$ ,  $Prod$ , and  $M$  which are defined, respectively, by

$$\begin{aligned} W(a, b) &= \max\{a + b - 1, 0\}, \\ Prod(a, b) &= ab, \\ M(a, b) &= \min(a, b). \end{aligned}$$

**Definition 2.1.** ([1]) A PN space is a quadruple  $(V, \nu, \tau, \tau^*)$  where  $V$  is a real linear vector space,  $\tau$  and  $\tau^*$  are continuous triangle functions and the probabilistic norm  $\nu$  is a mapping from  $V$  into  $\Delta^+$  such that, for all  $p, q$  in  $V$ , the following conditions hold (we use  $\nu_p$  instead of  $\nu(p)$ ):

- (N1)  $\nu_p = \varepsilon_0$ , if and only if  $p = \vartheta$  (where  $\vartheta$  is a null vector in  $V$ ),
- (N2)  $\nu_{-p} = \nu_p$
- (N3)  $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ ,
- (N4)  $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$  for every  $\lambda \in [0, 1]$ .

Since  $\tau$  is continuous, the system of strong neighborhoods  $\{N_p(t) | p \in V, t > 0\}$ , where

$$N_p(t) = \{q \in V | \nu_{p-q}(t) > 1 - t\},$$

determines a topology on  $V$ , called the *strong topology*.

**Definition 2.2.** ([6]) Let  $(V, \nu, \tau, \tau^*)$  be a PN space and  $A$  is a nonempty subset of  $V$ . The probabilistic diameter of  $A$ ,  $R_A$  is defined as

$$R_A(x) = \begin{cases} l^- \varphi_A(x), & x \in [0, +\infty); \\ 1, & x = +\infty, \end{cases}$$

where  $l^- f(x)$  denotes the left limit of the function  $f$  at  $x$  and

$$\varphi_A(x) = \inf_{p \in A} \nu_p(x)$$

As a consequence of Definition 2.2, we have  $\nu_p \geq R_A$ .

### 3. MAIN RESULTS

Now we propose the following *strong  $t$ -continuity* in PN spaces.

**Definition 3.1.** In a PN space  $(V, \nu, \tau, \tau^*)$ , a map  $f: V \rightarrow V$  is said to be strong  $t$ -continuous, ( $t > 0$ ) provided for each  $p \in V$  admits a strong  $t'$ -neighborhood  $N_p(t')$ ,  $t' > 0$  such that

$$R_{f(N_p(t'))}(t) > 1 - t.$$

The following theorem is a direct consequence of the Definition 3.1 above.

**Theorem 3.1.** Let  $(V, \nu, \tau, \tau^*)$  be a PN space such that  $\tau = \tau_M$  and  $A \subseteq V$ . Let  $f: A \rightarrow A$  be a strong  $t$ -continuous mapping where  $t > 0$ , then for every distinct  $p, q \in A$ ,

$$\nu_{f(p)-f(q)}(t) > 1 - t.$$

**Proof** Since  $f$  is a strong  $t$ -continuous mapping, for  $p, q \in A$ , there exist strong neighborhoods  $X = N_p(t')$ ,  $Y = N_q(t'')$  for every  $t', t'' > 0$  such that

$$\nu_{f(p)}(t) \geq R_{f(X)}(t) > 1 - t$$

and

$$\nu_{f(q)}(t) \geq R_{f(Y)}(t) > 1 - t$$

It follows that,

$$\begin{aligned} \nu_{f(p)-f(q)}(t) &\geq \tau_M(\nu_{f(p)}, \nu_{f(q)})(t) \\ &= \sup_{s \in (0,1)} M(\nu_{f(p)}(st), \nu_{f(q)}(t-st)) \\ &\geq M(\nu_{f(p)}(t), \nu_{f(q)}(t)) \\ &> \min(1-t, 1-t) = 1-t. \end{aligned}$$

The existence and uniqueness of fixed points for classes of mappings on PN spaces depending on some kind of continuity assumption. Without this assumption, the theorem could be not valid; it needs not be completely false, though, see [4] and [5].

The discontinuous of a self mapping can be interpreted several ways. Here, we consider the jump discontinuity, i.e.,  $\lim_{x \rightarrow a} f(x) \neq f(a)$ . For this purpose, we propose the following measure of discontinuity in PN spaces (a generalization of the measure of discontinuity in normed spaces, see [5]).

**Definition 3.2.** Let  $(V, \nu, \tau, \tau^*)$  be a PN space,  $A \subseteq V$  be precompact and  $f: A \rightarrow A$  any map. We define the following *probabilistic measure of discontinuity*:

$$\psi_f(t) = \inf_{p \in A} \liminf_{\delta \rightarrow 0} \inf_{q \in N_p(\delta)} \nu_{f(p)-f(q)}(t), t > 0$$

.

It can be noted that if  $f$  is strong  $t$ -continuous self mapping then  $\psi_f = \varepsilon_0$ .

Any (discontinuous) function  $f: A \rightarrow A$ , can be transformed into a continuous point-to-set mapping  $T_f$  as follows: for  $p \in A$ , we set

$$T_f(p) = \left\{ f(p_i) : \lim_{i \rightarrow \infty} \nu_{p_i-p} = \varepsilon_0 \right\}$$

,

i.e., the set  $T_f(p)$  assigned to a point  $p \in A$  includes all accumulation points of sequences  $\{f(p_i) : \lim_{i \rightarrow \infty} \nu_{p_i-p} = \varepsilon_0\}$ . Whence the point-to-set mapping  $p \xrightarrow{T} \text{con}T_f(p)$  (where  $\text{con}T_f(p)$  is convex hull of set  $T_f(p)$ ) is upper semi-continuous and may be interpreted as 'continuoufication' or smooting of the discontinuous original function  $f$ .

**Lemma 3.1.** Let  $(V, \nu, \tau, \tau^*)$  be a PN space,  $A \subseteq V$  be precompact and  $f: A \rightarrow A$  any map. Let  $q_i \rightarrow p^*$  be a convergent sequence in  $A$  and suppose for every  $i$ , there exists a convergent sequence  $p_{ij} \rightarrow p_i^*$  in  $A$  with

$$\lim_{j \rightarrow \infty} \nu_{f p_{ij}-q_i} = \varepsilon_0$$

.

Suppose further that  $p_i^* \rightarrow p^*$  for some  $p^* \in V$ . Then there exists a sequence  $r_j \rightarrow p^*$  in  $A$  such that

$$\lim_{j \rightarrow \infty} \nu_{f r_j-p^*} = \varepsilon_0$$

**Proof** By hypothesis, we have

$$\lim_{i \rightarrow \infty} \nu_{q_i-p^*} = \varepsilon_0, \lim_{i \rightarrow \infty} \nu_{p_i^*-p^*} = \varepsilon_0$$

and

$$\lim_{i \rightarrow \infty} \nu_{p_{ij}-p^*} = \varepsilon_0$$

. Let  $r_k = p_{i_k j_k}$ . By (PN2) and (PN3) we have,

$$\begin{aligned} \nu_{r_k - p^*} &\geq \tau(\nu_{r_k - p_{i_k}^*}, \nu_{p_{i_k}^* - p^*}) \\ &\geq \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0 \end{aligned}$$

and

$$\begin{aligned} \nu_{f r_k - q^*} &\geq \tau(\nu_{f r_k - q_{i_k}}, \nu_{q_{i_k} - q^*}) \\ &\geq \tau(\varepsilon_0, \varepsilon_0) = \varepsilon_0 \end{aligned}$$

as  $k \rightarrow \infty$ . Thus  $r_k \rightarrow p^*$  and  $\lim_{k \rightarrow \infty} \nu_{f r_k - q^*} = \varepsilon_0$ , as required.

**Lemma 3.2.** Let  $(V, \nu, \tau, \tau^*)$  be a PN space,  $A$  be a precompact and convex subset of  $V$  and let  $f: A \rightarrow A$  be any map. We define a set valued map  $T_f(p)$  on  $A$ :

$$T_f(p) = \{q \in A \mid \exists p_i \rightarrow p, p_i \neq p, \lim_{i \rightarrow \infty} \nu_{f p_i - q} = \varepsilon_0\}$$

If  $V = \{p\}$  we set  $T_f(p) = \{p\}$ . Then there exists a point  $p^* \in A$  such that  $p^* \in \text{con} T_f(p^*)$ .

**Proof** Let  $F(p) = \text{con} T_f(p)$ . We show that the graph  $G = \{(p, q) \mid p \in A, q \in F(p)\}$  is closed subset of  $A \times A$ . Let  $(p_i, q_i) \rightarrow (p, q)$  be a convergent sequence in  $A \times A$  with  $q_i \in F(p_i)$ . If  $p_i$  equals  $p$  infinitely often then it is trivial. Without loss of generality we assume  $p_i \neq p, \forall i = 1, 2, \dots, n$ . By theorem of Caratheodory, since  $q_i \in \text{con} T_f(p_i)$ , we can for every  $q_i$  choose  $n+1$  points  $p_{i_0}, \dots, p_{i_n} \in T_f(p_i)$  and scalars  $\lambda_{i_0}, \dots, \lambda_{i_n} \geq 0$  with  $\sum \lambda_{i_j} = 1$ , such that

$$q_i = \sum_{j=1}^n \lambda_{i_j} p_{i_j}$$

Since  $\lambda_i = (\lambda_{i_0}, \dots, \lambda_{i_n})$  is contained in the compact standard simplex  $S_n$  and the  $p_{i_j}$  are contained in the precompact set  $A$  we can consider a subsequence  $\{q_{i_j}\} \subset \{q_i\}$  with  $\nu_{q_{i_j} - q} \xrightarrow{w} \varepsilon_0$  such that the corresponding subsequences  $\{\lambda_{k_i}\}_{i=1}^\infty, j = 1, \dots, n$  and  $\{p_{k_i}\}_{i=1}^\infty, j = 1, \dots, n$ , converges to points  $\lambda_k$  and  $p_k$  respectively. It follows that  $q$  can be written as  $q = \sum_{k=0}^n \lambda_k p_k$ . Thus,  $q \in \text{con}\{p_0, \dots, p_n\}$ . It remains to show that  $p_k \in T_f(p)$  for  $k = 0, \dots, n$ . For fixed  $k$  we set  $s_i = p_{k_i}$  and  $s^* = p_k$ . Since  $s_i \in T_f(p_i)$  there exists a sequence  $t_{ij} \rightarrow p_i, t_{ij} \neq p_i$  such that  $\nu_{f(t_{ij}) - s_i} \xrightarrow{w} \varepsilon_0$ . Furthermore,  $p_i \rightarrow p$ . Thus from Lemma 3.1, there exists a sequence  $w_j \rightarrow p$  with  $\nu_{f(w_j) - s^*} \xrightarrow{w} \varepsilon_0$ . Also  $p_i \neq p, \forall i = 1, 2, \dots$ . From the definition of  $T_f$ , it now follows that  $s^* \in T_f(p)$ , i.e.,  $p_k \in T_f(p)$ .

**Remarks 3.1.** Since the extremal points of  $\text{con} T_f(p)$  are contained in  $T_f(p)$  we conclude that  $\psi_f$  is equal to the infimum of the probabilistic radius

of  $T_f(p)$  for  $p$  in  $V$ , in other words:

$$\psi_f = \min_{p \in V} R_{T_f(p)} = \min_{p \in V} \min_{q \in T_f(p)} \nu_{f(p)-q}$$

The theorem below represents the probabilistic generalization of the famous Kakutani's fixed point theorem.

**Theorem 3.2.** Let  $A$  be a nonempty, precompact, and convex subset of a PN-space  $(V, \nu, \tau, \tau^*)$  and let  $f: A \rightarrow A$  be any map. We set

$$\{(p, q) | p \in A, q \in T_f(p), \nu_{p-q} \xrightarrow{w} \varepsilon_0\} \subseteq V \times V$$

is closed. Then there is a point  $p^* \in V$  such that  $p^* \in T_p(p^*)$ .

Now, we consider our main theorem which shows that for any discontinuous function  $f$  there exists at least an "approximate fixed point", i.e., a point  $p^*$  whose image,  $f(p^*)$ , is no further away from  $p^*$  than by a certain degree of measure of discontinuity for  $f$ .

**Theorem 3.3.** Given a PN space  $(V, \nu, \tau, \tau^*)$ . Let  $A \subseteq V$  be a precompact and convex and  $f: A \rightarrow A$  any map, then there exists a point  $p^* \in A$  such that  $\nu_{f(p^*)-p^*} \geq \psi_f$ .

**Proof** By Lemma 3.2, there exists  $p^* \in \text{con}T_f(p^*)$ . Hence

$$\nu_{f(p^*)-p^*}(t) \geq \inf_{q \in T_f(p^*)} \nu_{f(p^*)-q}(t) \geq \inf_{p \in A} \inf_{q \in T_f(p)} \nu_{f(p)-q}(t) = \psi_f(t)$$

for every  $t > 0$ , i.e.,  $\nu_{f(p^*)-p^*} \geq \psi_f$  as required.

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